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Classical cellular convection with a spatial heat source

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This paper considers the problem of the stability of an infinite horizontal layer of a viscous fluid which loses heat throughout its volume at a constant rate. The variation of the critical Rayleigh number, R_{crit}, and the cell aspect ratio, a, with the rate of heat loss, is calculated with two sets of boundary conditions corresponding to two free and two rigid boundaries. In both cases we find that, as the rate of heat loss increases, R_{crit} decreases, showing that the layer becomes more unstable, and a increases, showing that the cells become narrower. We also consider the possibility that a double layer of cells is formed for large values of the rate of heat loss, by the stable layer at the top, and find that this does not occur while the temperature of the upper surface of the layer is less than that of the lower.

1. Introduction

Ray & Scorer (1963, chapter 4) considered the problem of the stability of a horizontal layer of a viscous fluid which loses heat throughout its volume while the temperatures of the bounding surfaces remain fixed. Two sets of boundary conditions were used, one in which both boundaries were rigid, and another in which both boundaries were assumed to be free surfaces. The heat loss was assumed in one case to be constant throughout the volume, and in another case to be periodic in z. When the heat loss was constant, which is the only case considered here, a non-dimensional number Q was used to express the volume heat loss as a multiple of the mean heat flux, and the critical Rayleigh number and associated value of the cell aspect ratio a were calculated for various values of Q. It was found that in each case the critical Rayleigh number decreased with increasing Q, but that the aspect ratio a increased with increasing Q when the boundaries were free, and decreased with increasing Q when the boundaries were rigid. It was not clear why the boundary conditions should affect the results in such a way, so the work was re-examined and found to contain algebraic errors. These have now been corrected and the results are given here.

The determination of critical Rayleigh numbers for this problem when the boundaries are rigid was subsequently carried out by Sparrow, Goldstein & Jonsson (1964), using a power series expansion, and by Debler (1965), using the analogy between this problem and that of the stability of flow between rotating cylinders (Chandrasekhar 1961). The case of the stability of a layer with two free boundaries does not appear to have been treated, although it is more likely to be of importance in the study of the stability of layers of air in the atmosphere.

Here we have calculated critical values of R and a for various values of Q in both these problems and compared the results. We were also interested in the possibility that a two-layer flow pattern might be produced as the upper layers become more stable because they are eventually driven by the viscous drag of the motion in the unstable lower layers. It is shown that the motion remains in the form of a single cell, at least so long as the temperature of the upper surface is less than that of the lower. The method of solution of the equations is similar to that used by Chandrasekhar (1961) to solve the problem of flow between rotating cylinders.

2. Development of the equations

We consider a layer of viscous fluid of infinite horizontal extent and of finite depth h. A co-ordinate system $Oxyz$ is chosen, having Oz vertical and Ox, Oy in a horizontal plane. The lower and upper boundaries of the layer, at $z = 0, h$, are maintained at the constant temperatures T_0 and T_1 respectively, and throughout the volume of the fluid heat is lost at a constant rate. Since the equations contain only gradients of temperature this heat loss may be thought of as heat absorption by a layer becoming steadily warmer.

The governing equations of fluid flow, heat conduction, and mass conservation are

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{g}\rho - \nabla p + \mu\nabla^2\mathbf{v}, \tag{2.1}$$

$$\frac{DT}{Dt} = -q + k\nabla^2 T, \tag{2.2}$$

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0, \tag{2.3}$$

where ρ is the density of the fluid; $\mathbf{v} = (u, v, w)$ is the fluid velocity; $\mathbf{g} = (0, 0, -g)$ is gravity; p is the pressure in the fluid; μ the dynamic viscosity; T the absolute temperature; k the thermal conductivity; and q the heat lost within the fluid per unit volume per unit time.

With a constant coefficient of expansion α, we have

$$\rho = \rho_0(1 - \alpha(T - T_0)), \tag{2.4}$$

where ρ_0 and T_0 are constants.

In the equilibrium state heat transport is by conduction alone and there are no velocities in the fluid. The solution of equations (2.1)–(2.4) is easily obtained in this case, and, if we make the assumption that the mean state after the onset of slow convection is the same as the equilibrium state, we have for the mean state variables (denoted by an overbar)

$$\bar{\mathbf{v}} = 0, \tag{2.5}$$

$$\bar{p} = \mathbf{g}\bar{\rho}, \tag{2.6}$$

$$\nabla^2\bar{T} = \frac{q}{k}, \tag{2.7}$$

$$\bar{\rho} = \rho_0(1 - \alpha(\bar{T} - T_0)). \tag{2.8}$$

The mean temperature $\bar{T}$ is assumed to depend only on z, so we can integrate (2.7) to give

$$\bar{T} = \frac{q}{2k}(z^2 - hz) - \frac{\Delta T}{h} z + T_0, \tag{2.9}$$

where $\Delta T = T_0 - T_1 > 0$, since this is the solution which gives $\bar{T} = T_0, T_1$ at $z = 0, h$. Writing

$$p = \bar{p} + p', \quad T = \bar{T} + T', \quad \rho = \bar{\rho} + \rho', \quad \mathbf{v} = \mathbf{v}, \tag{2.10}$$

where p', T', ρ' and $\mathbf{v}$ are small perturbations with respect to which the equations are linearized using the Boussinesq approximation, we obtain

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0}\nabla p' + \nu \nabla^2 \mathbf{v} + \frac{\mathbf{g}\rho'}{\rho_0}, \tag{2.11}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{2.12}$$

$$\frac{\partial T'}{\partial t} - k\nabla^2 T' = -w\frac{d\bar{T}}{dz}, \tag{2.13}$$

$$\rho' = -\rho_0 \alpha T', \tag{2.14}$$

where ν is the kinematic viscosity. These equations are similar to those obtained by Pellew & Southwell (1940) except that in this case $d\bar{T}/dz$ is not constant.

Eliminating ρ' and $\bar{T}$ we have

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{g}\alpha T' - \frac{1}{\rho_0}\nabla p' + \nu\nabla^2 \mathbf{v}, \tag{2.15}$$

$$\left[\frac{\partial}{\partial t} - k\nabla^2\right] T' = -\left[\frac{qz}{k} - \frac{qh}{2k} - \frac{\Delta T}{h}\right] w, \tag{2.16}$$

leading to

$$\left[\frac{\partial}{\partial t} - \nu\nabla^2\right]\nabla^2 w = g\alpha \nabla_1^2 T', \tag{2.17}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and finally to an equation for w alone,

$$\left[\frac{\partial}{\partial t} - k\nabla^2\right]\left[\frac{\partial}{\partial t} - \nu\nabla^2\right]\nabla^2 w = -g\alpha\left[\frac{q}{2k}(2z - h) - \frac{\Delta T}{h}\right]\nabla_1^2 w. \tag{2.18}$$

We now assume a separable solution of the form

$$w(x, y, z; t) = w(z) f(x, y)\, e^{\sigma t}, \tag{2.19}$$

where

$$\nabla_1^2 f(x,y) + \frac{a^2}{h^2} f(x,y) = 0,$$

in which a is the aspect ratio of the cells (Pellew & Southwell 1940). Hence

$$\nabla_1^2 w = -\frac{a^2}{h^2} w. \tag{2.20}$$

Setting

$$\zeta = \frac{z}{h}, \quad D = \frac{\partial}{\partial \zeta},$$

we have

$$\nabla^2 w = \nabla_1^2 w + \frac{\partial^2 w}{\partial z^2} = \frac{1}{h^2}(D^2 - a^2) w, \tag{2.21}$$

so that (2.18) becomes

$$\left[\sigma-\frac{k}{h^2}(D^2-a^2)\right]\left[\sigma-\frac{\nu}{h^2}(D^2-a^2)\right]\frac{(D^2-a^2)}{h^2}w(z) = g\alpha\left[\frac{qh}{2k}(2\zeta-1)-\frac{\Delta T}{h}\right]\frac{a^2}{h^2}w(z). \tag{2.22}$$

On the usual assumption that the critical temperature gradient corresponds to $\sigma = 0$, we have

$$(D^2-a^2)^3 w = \frac{g\alpha h^4}{k}\left[-\frac{\Delta T}{h}+\frac{qh}{2k}(2\zeta-1)\right]a^2 w,$$

i.e.

$$(D^2-a^2)^3 w = -(R+Q(1-2\zeta))a^2 w, \tag{2.23}$$

where

$$R = \frac{g\alpha h^3 \Delta T}{k\nu}, \quad \text{the Rayleigh number,}$$

and

$$Q = \frac{g\alpha q h^5}{2k^2\nu} = R\frac{qh^3}{2kh\Delta T} = \frac{R\times\text{heat lost}}{2\times\text{mean heat flux}};$$

$$\begin{aligned}\text{mean heat flux} &= \tfrac{1}{2}\,(\text{heat flux in at bottom} + \text{heat flux out at top})\\ &= \text{constant.}\end{aligned}$$

Q gives the heat loss as a multiple of the mean heat flux, the factor 2 being introduced for convenience. Equation (2.23) is the same as that obtained by Ray & Scorer.

3. The problem of a layer with free boundaries

(i) *Boundary conditions*

With fixed, stress-free boundaries at $\zeta = 0, 1$ we have

$$w = 0, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \quad (\zeta = 0, 1), \tag{3.1}$$

whence

$$\frac{\partial^2 u}{\partial x\,\partial z}+\frac{\partial^2 v}{\partial y\,\partial z} = 0 = -\frac{\partial^2 w}{\partial z^2} \quad (\zeta = 0, 1);$$

i.e.

$$D^2 w = 0 \quad (\zeta = 0, 1). \tag{3.2}$$

The thermal boundary condition $\bar{T}$ = constant implies $T' = 0$, and can be expressed in terms of w by means of (2.18), giving

$$\left[\frac{\partial}{\partial t}-\nu\nabla^2\right]\nabla^2 w = 0 \quad (\zeta = 0, 1);$$

i.e.

$$\nabla^4 w = 0 \quad \text{or} \quad D^4 w = 0 \quad (\zeta = 0, 1). \tag{3.3}$$

From the governing equation and (3.1)–(3.3) we thus obtain

$$D^6 w = D^8 w = \ldots = D^{2n} w = 0 \quad (\zeta = 0, 1),$$

and so

$$w = D^2 w = D^4 w = \ldots = D^{2n} w = 0 \quad (\zeta = 0, 1). \tag{3.4}$$

(ii) *Solution of the equation*

We have to solve (2.23) with the boundary conditions (3.4). Writing

$$\psi = a^2(R+Q)w, \tag{3.5}$$

equation (2.23) becomes
$$(D^2-a^2)^3 w = -(1+C\zeta)\psi, \tag{3.6}$$
where
$$C = -\frac{2Q}{R+Q}.$$
We assume a solution
$$w = \sum_{n=1}^{\infty} \Gamma_m w_m, \tag{3.7}$$
$$\psi = \sum_{m=1}^{\infty} \Gamma_m \psi_m, \tag{3.8}$$
where Γ_m is a constant, and ψ_m, w_m satisfy (3.6) and the boundary conditions (3.4) for each m. If $Q = 0$ ($C = 0$) equation (3.6) for ψ_m, w_m and boundary conditions (3.4) are satisfied by $\psi_m = \sin m\pi\zeta$, $w_m = \sin m\pi(m^2\pi^2+a^2)^{-3}$, so we may usefully set $\psi_m = \sin m\pi\zeta$ in equation (3.6), giving
$$(D^2-a^2)^3 w_m = -(1+C\zeta)\sin m\pi\zeta, \tag{3.9}$$
which is easily integrated to give
$$w_m = \frac{1}{(m^2\pi^2+a^2)^3}\Big\{(A_m+B_m\zeta+C_m\zeta^2)\sinh a\zeta+(D_m+E_m\zeta+F_m\zeta^2)\cosh a\zeta + (1+C\zeta)\sin m\pi\zeta+\frac{6m\pi C}{m^2\pi^2+a^2}\cos m\pi\zeta\Big\}. \tag{3.10}$$
Applying the boundary conditions (3.4) to w_m we find that the constants $A_m, \ldots, F_m$ are
$$\left.\begin{aligned}
A_m &= \frac{m\pi C}{a^2\sinh a}\Big\{\Big[\frac{m^2\pi^2+9a^2}{4a}+\frac{(m^2\pi^2+a^2)}{2\sinh a}\cosh a\Big]\Big[\frac{1-(-1)^m\cosh a}{\sinh a}\Big] \\
&\quad +(-1)^m\frac{(m^2\pi^2+a^2)}{4}+\frac{6a^2}{m^2\pi^2+a^2}[\cosh a-(-1)^m]\Big\},\\
B_m &= \frac{m\pi C}{4a^3}(m^2\pi^2+9a^2),\\
C_m &= \frac{m\pi C}{4a^2\sinh a}(m^2\pi^2+a^2)(\cosh a-(-1)^m),\\
D_m &= -\frac{6m\pi C}{m^2\pi^2+a^2},\\
E_m &= -\frac{m\pi C}{2a^2\sinh a}\Big\{\frac{(m^2\pi^2+a^2)}{\sinh a}(1-(-1)^m\cosh a) \\
&\quad +\frac{(m^2\pi^2+9a^2)}{2a}(\cosh a-(-1)^m)\Big\},\\
F_m &= -\frac{m\pi C}{4a^2}(m^2\pi^2+a^2).
\end{aligned}\right\} \tag{3.11}$$
Substituting from (3.7) and (3.8) into (3.5) we have
$$\sum_{m=1}^{\infty}\Gamma'_m\Big\{(m^2\pi^2+a^2)^3\sin m\pi\zeta-a^2(R+Q)\Big\{(A_m+B_m\zeta+C_m\zeta^2)\sinh a\zeta + (D_m+E_m\zeta+F_m\zeta^2)\cosh a\zeta+(1+C\zeta)\sin m\pi\zeta+\frac{6m\pi C}{m^2\pi^2+a^2}\cos m\pi\zeta\Big\}\Big\} = 0, \tag{3.12}$$

where
$$\Gamma'_m = \frac{\Gamma_m}{(m^2\pi^2+a^2)^3}.$$

Multiplying by sin $n\pi\zeta$ and integrating from 0 to 1 gives a set of homogeneous linear equations in the Γ'_m, and for a non-trivial solution the determinant of coefficients must vanish, giving rise to the secular equation

$$\|\tfrac{1}{2}(n^2\pi^2+a^2)^3\delta_{mn}-a^2(R+Q)(m:n)\| = 0, \tag{3.13}$$

where
$$(m:n) = \int_0^1 (m^2\pi^2+a^2)^3 w_m \sin n\pi\zeta\, d\zeta. \tag{3.14}$$

This is the equation obtained by Ray & Scorer, but an algebraic error was made in the calculation of the constants A_m and E_m in (3.11).

4. The problem of a layer with rigid boundaries

(i) *Boundary conditions*

With fixed, rigid boundaries at $\zeta = 0, 1$ we have

$$u = v = w = 0 \quad (\zeta = 0, 1), \tag{4.1}$$

$$\text{whence } -\frac{\partial w}{\partial z} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \quad (\zeta = 0, 1),$$

i.e.
$$Dw = 0 \quad (\zeta = 0, 1). \tag{4.2}$$

As in §3(i) the thermal boundary condition is

$$\nabla^4 w = 0 \quad (\zeta = 0, 1),$$

which we write in the form

$$(D^2-a^2)^2 w = 0 \quad (\zeta = 0, 1). \tag{4.3}$$

So finally
$$w = Dw = (D^2-a^2)^2 w = 0 \quad (\zeta = 0, 1). \tag{4.4}$$

(ii) *Solution of the equation*

In this case we set
$$(D^2-a^2)^2 w = \psi \tag{4.5}$$
and (2.23) becomes
$$(D^2-a^2)\psi = -(R+Q(1-2\zeta))a^2 w. \tag{4.6}$$

As in §3 we assume a solution

$$w = \sum_{m=1}^{\infty} \Gamma_m w_m, \tag{4.7}$$

$$\psi = \sum_{m=1}^{\infty} \Gamma_m \psi_m, \tag{4.8}$$

where Γ_m is a constant and ψ_m, w_m satisfy equation (4.5) and the boundary conditions (4.4) for each m. In this case

$$\psi_m = \sin m\pi, \quad w_m = \frac{\sin m\pi}{m^2\pi^2+a^2}$$

satisfy (4.5) and the boundary conditions (4.4) for all Q, and we again set $\psi_m = \sin m\pi\zeta$, equation (4.5) becoming

$$(D^2-a^2)^2 w_m = \sin m\pi\zeta, \tag{4.9}$$

which integrates to give

$$w_m = \frac{1}{(m^2\pi^2+a^2)^2}\left\{(A_m+B_m\zeta)\sinh a\zeta+(C_m+D_m\zeta)\cosh a\zeta+\sin m\pi\zeta\right\}. \quad (4.10)$$

Here we have only four arbitrary constants since (4.9) is a fourth-order equation. These constants are obtained from the boundary conditions

$$w_m = Dw_m = 0 \quad (\zeta = 0, 1),$$

the third condition, $$(D^2-a^2)^2 w_m = 0 \quad (\zeta = 0, 1),$$

being satisfied automatically because of (4.9). The values of the constants $A_m, \ldots, D_m$ are

$$\left.\begin{aligned} A_m &= \frac{m\pi}{a^2-\sinh^2 a}\{(-1)^{m+1}\sinh a - a\},\\ B_m &= \frac{m\pi}{a^2-\sinh^2 a}\{(-1)^{m+1}(a\cosh a-\sinh a)+a-\sinh a\cosh a\},\\ C_m &= 0,\\ D_m &= \frac{-m\pi}{a^2-\sinh^2 a}\{(-1)^{m+1}a-\sinh a\}\sinh a. \end{aligned}\right\} \quad (4.11)$$

Substituting the series for ψ and w into equation (4.6) we have

$$\sum_{m=1}^{\infty}\Gamma'_m\{(m^2\pi^2+a^2)^3\sin m\pi\zeta - a^2(R+Q(1-2\zeta))\{(A_m+B_m\zeta)\sinh a\zeta + D_m\zeta\cosh a\zeta+\sin m\pi\zeta\}\} = 0, \quad (4.12)$$

where $$\Gamma'_m = \frac{\Gamma_m}{(m^2\pi^2+a^2)^2}.$$

The secular equation obtained from this in the manner described in (3) is

$$\|\tfrac{1}{2}(n^2\pi^2+a^2)^3\delta_{mn}-(m:n)\| = 0, \quad (4.13)$$

where $$(m:n) = \int_0^1 (m^2\pi^2+a^2)^2 w_m a^2(R+Q(1-2\zeta))\sin n\pi\zeta\, d\zeta. \quad (4.14)$$

In this case the equation (4.13) and constants (4.11) are the same as those obtained by Ray & Scorer, but in their work an error was made in calculating the approximations to the secular equation described in § (5).

5. Calculation of an approximate solution

An approximation to the solution can be obtained by setting the determinant formed by the elements in the first n rows and columns of the secular determinant equal to zero, successive approximations being obtained as n increases. Convergence is good for small Q, but becomes poorer as Q increases. This procedure is equivalent to taking the first n terms in the expansions for ψ and w. The calculation is done for various values of Q, in each case the value of R which corresponds to a particular value of a being found by solving an algebraic equation. The minimum value of R with respect to a is then calculated for each Q.

Sparrow *et al.* (1964) calculated the critical Rayleigh number subject to the condition N_s = constant, where N_s is equivalent to Q/R in the present work. This gives the same value for R_{crit} as our calculation since we see from

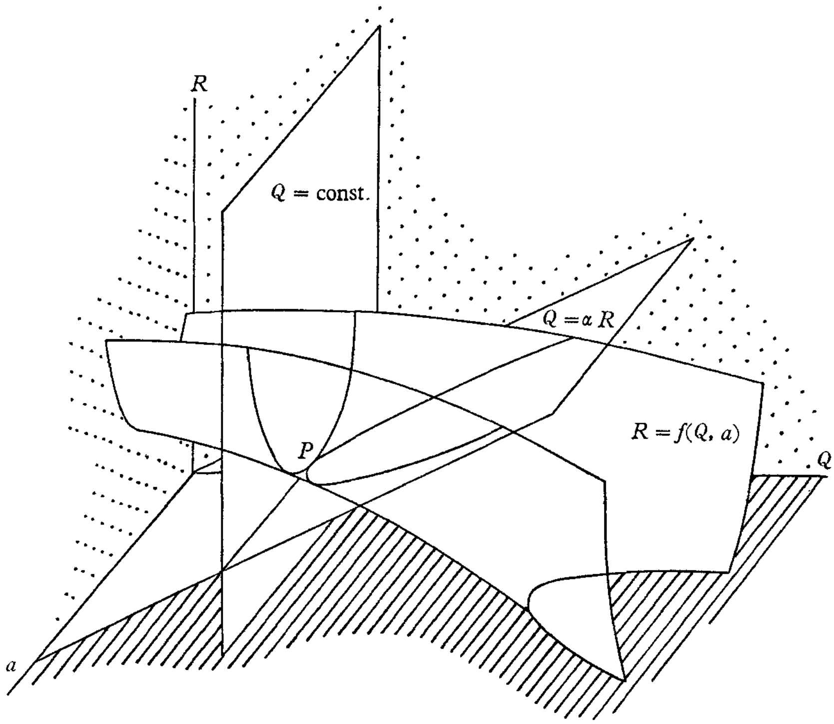

FIGURE 1. The surface $R = f(Q,a)$ cut by planes Q = const. and $Q = \alpha R$. The point P represents R_{crit}.

figure 1 that the surface $R = f(a, Q)$ is such that the curve formed by the intersection of the surface with the plane Q = constant has a unique minimum, R_{crit}, and that the plane $Q = \alpha R$ through this minimum point does not meet the surface again for $0 < R < R_{crit}$, so that R_{crit} is a minimum in this plane also.

The approximate solutions obtained from the first few terms of equations (2.7), (4.7) can be used to discover whether the motion within the layer is in the form of a single or a double cell. When the temperature profile is parabolic and there is a temperature minimum within the fluid, stable fluid is lying above unstable fluid. If the stable layer were very thick we should expect the convection to penetrate only a small distance into the stable layer, so that the vertical velocity would be reduced to zero at some point within the layer. We investigate the presence or absence of this zero by calculating the values of $Dw|_0$ and $Dw|_1$ in the case of free boundaries, and $D^2w|_0$ and $D^2w|_1$ in the case of rigid boundaries, to see whether or not Dw or D^2w change sign within the layer.

6. Heat generated within the fluid

If heat is generated within the fluid Q will be replaced by $-Q$, and instead of (2.23) we have

$$(D^2-a^2)^3 w = -(R-Q(1-2\zeta))\,a^2 w. \tag{6.1}$$

The boundary conditions remain the same, (3.4) or (4.4). The substitution $\zeta = 1-\eta$ transforms (6.1) to

$$(D^2-a^2)^3 w = -(R+Q(1-2\eta))a^2 w, \tag{6.2}$$

where now $D = \partial/\delta\eta$. This is the same as (2.23). Applying $\zeta = 1-\eta$ to the boundary conditions (3.4) or (4.4) simply interchanges the conditions at the upper and lower boundaries, and, since these are the same, the boundary conditions do not change under this transformation. Hence the two problems are identical under the transformation $\zeta = 1-\eta$.

Support for this argument is obtained from the secular equation. In the algebraic approximations used, Q appears in even powers only, so the same result is obtained for the critical value of R regardless of the sign of Q.

7. Results

Figure 2 shows the behaviour of R_{crit} and a with Q in both the cases considered. The figures for $n = 2, 3, 4$ and 5 are given in tables 1 and 2. We see that in both cases as Q increases R_{crit} decreases; i.e. the layer becomes less stable.

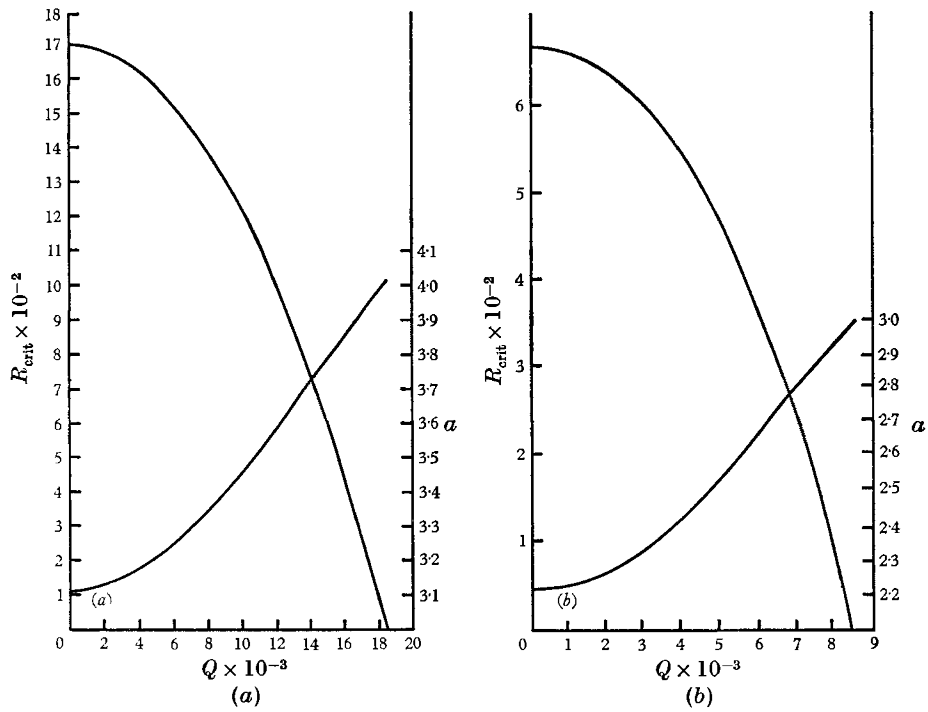

FIGURE 2. Variation of R_{crit} and a with Q, (a) for rigid boundaries, (b) for free boundaries.

This agrees with Ray & Scorer, but, whereas they found a linear dependence of R_{crit} on Q, we find, as was mentioned in (6), that R_{crit} depends on even powers of Q only, and so on Q^2 at least. It should be noted that the Rayleigh number used here is based on the temperature difference between the top and bottom of the layer, and so gives no measure of the actual temperature gradients within the layer unless Q is small. To obtain a measure of these gradients we must use a parameter based on the difference between the minimum temperature in the fluid and the temperature at one of the boundaries (Debler 1965; Sparrow *et al.* 1964).

We note that for small Q the critical Rayleigh number is very little different from its value at $Q = 0$, and does not begin to depart radically from this value until Q and R are of comparable magnitudes. This is to be expected, since, for

(a)

Q	R_{crit} $n = 2$	R_{crit} $n = 3$	R_{crit} $n = 4$
0	657·5	657·5	657·5
500	655·6	655·6	655·6
1000	649·9	649·9	649·9
2000	626·7	626·7	626·7
3000	586·9	586·9	586·9
4000	529·0	528·9	528·8
5000	451·1	450·7	450·6
6000	351·5	350·5	350·4
7000	229·2	227·3	227·2
8000	85·5	81·0	80·9

(b)

Q	a $n = 2$	a $n = 3$	a $n = 4$
0	2·22	2·22	2·22
500	2·22	2·22	2·22
1000	2·23	2·23	2·23
2000	2·28	2·28	2·28
3000	2·34	2·34	2·34
4000	2·44	2·44	2·44
5000	2·56	2·56	2·56
6000	2·70	2·70	2·70
7000	2·83	2·83	2·83
8000	2·95	2·97	2·97

(c)

	$R = 0$ $n = 2$	$R = 0$ $n = 3$	$R = 0$ $n = 4$
Q	8533	8497	8496
a	3·02	3·03	3·03

TABLE 1. Selection of results showing variation of R_{crit} and a with Q when the boundaries are free

$Q < R$, no minimum of temperature occurs within the layer, and the temperature gradients are little different from those in the linear case. When $Q > R$, however, a temperature minimum occurs within the layer, so that there may be very large temperature gradients.

Turning our attention to the behaviour of the aspect ratio a as Q increases, we see that it is very much the same in both cases. The aspect ratio has increased by about 30% when R is zero. This corresponds to a decrease in the horizontal dimensions of the cells.

In figure 3, the vertical velocity w is plotted, with the maximum being taken to be unity for all values of Q. There is no means of finding the absolute magnitude of w in this case since the equations allow an arbitrary constant multiplier. We see, however, that as Q increases the maximum vertical velocity occurs at a smaller value of ζ, i.e. lower in the layer. This means that the centre of circulation

(a)

	R_{crit}			
Q	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	1715·0	1707·9	1707·9	1707·8
1,000	1710·5	1703·5	1703·5	1703·3
2,000	1696·8	1690·2	1690·1	1689·9
4,000	1641·6	1636·7	1636·1	1636·0
6,000	1548·1	1546·1	1544·7	1544·5
8,000	1414·2	1416·3	1413·9	1413·6
10,000	1238·2	1245·3	1241·6	1241·4
12,000	1019·1	1031·6	1026·3	1026·2
14,000	756·9	774·6	767·5	767·6
16,000	453·6	475·0	466·3	466·7
18,000	112·1	134·7	124·4	125·5

(b)

	a			
Q	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	3·11	3·116	3·116	3·116
1,000	3·12	3·120	3·120	3·120
2,000	3·13	3·130	3·130	3·130
4,000	3·17	3·173	3·173	3·173
6,000	3·25	3·242	3·243	3·243
8,000	3·34	3·337	3·337	3·338
10,000	3·46	3·451	3·454	3·454
12,000	3·59	3·577	3·581	3·581
14,000	3·72	3·710	3·712	3·712
16,000	3·84	3·838	3·840	3·840
18,000	3·96	3·959	3·959	3·959

(c)

	$R = 0$			
	$n = 2$	$n = 3$	$n = 4$	$n = 5$
Q	1861·5	1873·4	1867·7	1867·4
a	3·989	4·001	3·998	4·000

TABLE 2. Selection of results showing variation of R_{crit} and a with Q when the boundaries are rigid

in the cell is nearer the lower surface for large Q. This is to be expected since the presence of stable fluid at the top of the layer will tend to decrease all velocities near the upper boundary. As a result of this decrease in velocity, less work is done by the buoyancy forces, and consequently less energy is available to drive the flow against viscosity, so the cell diameter becomes smaller, in agreement with the above result for a. From table 3 we see that, when the boundaries are rigid, D^2w changes sign an even number of times within the layer for all values of Q, implying that w has an odd number of stationary points in the layer. Since when $Q = 0$ there is only one stationary point, a maximum, and since the ratio

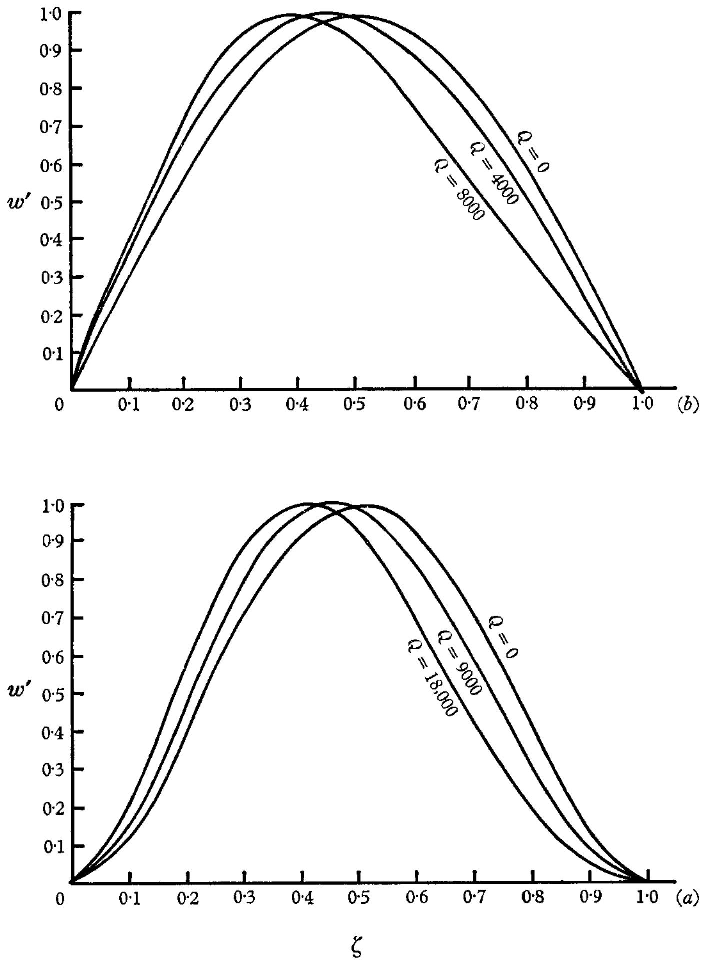

FIGURE 3. Normalized vertical velocity profile for selected values of Q, (a) for rigid boundaries, (b) for free boundaries.

Q	$D^2w\|_0$	$D^2w\|_1$	$D^2w\|_1/D^2w\|_0$
0	16·8	16·8	1
2,000	18·0	15·6	0·87
4,000	19·5	14·6	0·75
6,000	21·4	13·7	0·64
8,000	23·7	12·8	0·54
10,000	26·4	11·8	0·44
12,000	29·4	10·6	0·36
14,000	32·7	9·4	0·29
16,000	35·2	8·2	0·23
18,000	40·6	6·3	0·15

TABLE 3. Variation of $D^2w|_0$, $D^2w|_1$ and the ratio $D^2w|_1/D^2w|_0$ with Q when the boundaries are rigid

$D^2w|_1/D^2w|_0$ decreases steadily as Q increases, we conclude that there is only one stationary point for all Q such that $R > 0$.

Table 4 shows that Dw changes sign an odd number of times in the layer with free boundaries, so again there is an odd number of stationary points in the layer and, since the ratio $-Dw|_1/Dw|_0$ decreases as Q increases, we again conclude that there is a single stationary point in the layer.

Q	$Dw\|_0$	$Dw\|_1$	$-Dw\|_1/Dw\|_0$
0	3·14	−3·14	1
1000	1·3	−1·2	0·92
2000	0·86	−0·72	0·84
3000	0·66	−0·5	0·76
4000	0·55	−0·38	0·68
5000	0·49	−0·29	0·6
5000	0·45	−0·24	0·52
7000	0·43	−0·19	0·45
8000	0·42	−0·16	0·38

TABLE 4. Variation of $Dw|_0$, $Dw|_1$ and the ratio $-Dw|_1/Dw|_0$ with Q when the boundaries are free

In both cases there is no zero of the vertical velocity in the layer, so we have a single-cell circulation. This agrees with the result of Debler (1965) for rigid boundaries, that a double cell does not appear until $\eta > 0{\cdot}546$, where η represents the height of the temperature maximum above the lower surface of the fluid. In our case, η corresponds to the depth of the temperature minimum below the upper surface of the fluid, and is in no case greater than 0·5. We can obtain an estimate of the value of Q for which D^2w_1 or Dw_1 vanishes at R_{crit}, which will in this case be negative. We therefore conclude that if Q is less than this value (12,000, 32,000 for free and rigid boundaries respectively) there will be no double cells.

Comparing the two problems we see that the overall behaviour is very similar: the boundary conditions affect only the magnitudes of the critical Rayleigh number and the aspect ratio, and their effects on the vertical velocity profile are mainly confined to the parts of the layer near the boundaries.

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